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Technical Report No. 17

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A new E-field solution is presented for the electric current and electric charge induced on a perfectly conducting surface illuminated by an incident electromagnetic field. This solution is a moment solution to the electric field integral equation on the surface. The expansion functions consist of a set of functions suitable for expanding the magnetostatic current and a set of functions whose surface divergences are suitable for expanding the electrostatic charge. The testing functions are similar to the expansion functions. With these expansion and testing functions, the new E-field solution works well

20. ABSTRACT (continued)

with surfaces whose maximum dimension may be as small as $2 \times 10^{-15}$ wavelengths or as large as a few wavelengths. Previous E-field solutions begin to deteriorate when the maximum dimension of the surface falls below a few hundredths of a wavelength. The new E-field solution is applied to a conducting circular disk and a conducting sphere.

CONTENTS

## I. INTRODUCTION

Consider a perfectly conducting surface S placed in free space and illuminated by an incident electromagnetic field. The problem is to find the surface density $\underline{J}$ of electric current induced on S. A solution for $\underline{J}$ can be obtained by writing the electric field integral equation on S and then numerically solving this equation by means of the method of moments. This solution is called an E-field solution. E-field solutions have been applied to a rectangular bent plate [1], a surface of revolution [1]-[5] and a surface of arbitrary shape [6].

These E-field solutions are reasonably accurate in the resonance region. The resonance region is the range of frequencies for which the maximum dimension of the surface S is of the order of a wavelength. Unfortunately, an erroneous change in the slope of the radially directed electric current density near the center of a conducting circular disk of radius $0.02\lambda$ was reported in [5]. Here, $\lambda$ is the wavelength. For the dual problem of the circular aperture of radius $0.02\lambda$ in an infinite conducting plane, an erroneous change in the slope of the radially directed magnetic current density appears near the center of the aperture in [7, Fig. 6d]. In the E-field solution of [5], the erroneous change in the slope of the radially directed electric current density is more pronounced for the disk of radius $0.002\lambda$ than for the disk of radius $0.02\lambda$.

In general, all the E-field solutions in [1]-[6] begin to lose accuracy somewhere in the Rayleigh region and become worse as the frequency decreases. The Rayleigh region [8] is the range of frequencies for which the maximum dimension of S is much smaller than the wavelength. These E-field solutions fail in the low frequency portion of the Rayleigh region as shown by the following reasoning. When the frequency is sufficiently low, the magnetic vector potential contributions to the elements of the moment matrix are insignificant compared with the electric scalar potential contributions. As a result, the magnetic vector potential contributions are lost. The remaining scalar potential contributions depend only on $\nabla_s \cdot \underline{J}$. Here, $\nabla_s \cdot \underline{J}$ is the surface divergence of $\underline{J}$. Knowledge of $\nabla_s \cdot \underline{J}$ is not sufficient to determine $\underline{J}$. Therefore, the E-field solutions in [1]-[6] fail in the low frequency portion of the Rayleigh region.

It may be possible to obtain reasonably accurate values of $\underline{J}$ in the low frequency portion of the Rayleigh region in the following manner. Usually, there is at least one frequency that is high enough such that one of the E-field solutions for $\underline{J}$ in [1]-[6] is accurate but low enough such that $\underline{J}$ can be approximated by the first term of its Rayleigh series. The Rayleigh series for $\underline{J}$ is its low frequency expansion in non-negative integer powers of the frequency [8]. If there is such a frequency, the first term of the Rayleigh series for $\underline{J}$ can be extracted from one of the E-field solutions in [1]-[6]. Knowledge of the first term of its Rayleigh series amounts to knowledge of $\underline{J}$ in the low frequency portion of the Rayleigh region.

In many cases, it is of interest to obtain not only $\underline{J}$ but also the scattered field. The scattered field is the electromagnetic field produced by $\underline{J}$. In the resonance region, knowledge of $\underline{J}$ alone is sufficient to calculate the scattered field because the density $q_e$ of electric charge associated with $\underline{J}$ can be calculated from the equation of continuity

$$q_e = \frac{\nabla_s \cdot \underline{J}}{-j\omega} \tag{1}$$

In (1), $\omega$ is the angular frequency. However, in the low frequency portion of the Rayleigh region, knowledge of both $\underline{J}$ and $q_e$ is required in order to calculate the scattered field because $q_e$ can not be accurately calculated from $\underline{J}$ by means of (1). Accurate calculation of $q_e$ from $\underline{J}$ by means of (1) is usually not possible in the low frequency portion of the Rayleigh region, as shown by the following reasoning. Usually, $\underline{J}$ approaches a solenoidal vector function as the frequency approaches zero. As a result, $\nabla_s \cdot \underline{J}$ is so small that it can not be accurately calculated from $\underline{J}$. Therefore, accurate calculation of $q_e$ from $\underline{J}$ by means of (1) is usually not possible in the low frequency portion of the Rayleigh region. However, $q_e$ can be obtained in the same way that $\underline{J}$ was obtained in the preceding paragraph.

In the manner described in the previous two paragraphs, any one of the E-field solutions [1]-[6] for $\underline{J}$ can be extended into the low frequency portion of the Rayleigh region. The problem of the perfectly conducting surface S illuminated by an incident electromagnetic field in the low frequency portion of the Rayleigh region can be solved by other methods. For instance, the first term in the Rayleigh series for $\underline{J}$ can be obtained by solving a magnetostatic problem, and the first term in the Rayleigh series for $q_e$ can be obtained by solving an electrostatic problem. In most of the literature on low frequency electromagnetic scattering, Rayleigh series are constructed for the electric and magnetic fields rather than for the electric current and electric charge. For example, see [9]-[13].

In this paper, a new E-field solution is presented for the electric current $\underline{J}$ and the electric charge $q_e$ induced on the perfectly conducting surface S immersed in an incident electromagnetic field. In general, S consists of several surfaces that are disjoint from each other. Some of these surfaces may be open and others may be closed. The new E-field solution is similar to the E-field solutions in [1]-[6] because it is a moment solution of the electric field integral equation on S. However, the new E-field solution uses different expansion and testing functions. In the new E-field solution, the expansion functions consist of two sets of vector functions. The first set of vector functions is a suitable basis for expanding the magnetostatic current. The magnetostatic current is the first term in the Rayleigh series for $\underline{J}$. The second set of vector functions is such that the electric charges associated with them form a suitable basis for expanding the electrostatic charge. The electrostatic charge is the first term in the Rayleigh series for $q_e$. The testing functions in the new E-field solution are similar to the expansion functions. These expansion and testing functions render the moment matrix well-behaved as the frequency approaches zero.

So constructed, the new E-field solution should give accurate values of J and $q_e$ throughout both the Rayleigh region and the resonance region. Numerical results were obtained for conducting disks of radii $10^{-15}\lambda$ and $0.02\lambda$ and for conducting spheres of radii $10^{-15}\lambda$ and $0.02\lambda$. Each disk is excited by a plane wave propagating perpendicular to the plane of the disk. Each sphere is excited by a plane wave. These numerical results agree well with the known solutions for the electric current and electric charge on a small disk [14] and a sphere [15, Eq. (6-103)].

## II. THE MAGNETOSTATIC CURRENT AND THE ELECTROSTATIC CHARGE

Since some of the expansion functions for the new E-field solution will form a suitable basis for expanding the magnetostatic current and the surface divergences of the rest of the expansion functions will form a suitable basis for expanding the electrostatic charge, it is helpful to establish integral equations for the magnetostatic current and the electrostatic charge. Such integral equations can be obtained by substituting Rayleigh series for $\underline{J}$ and $\underline{E}^{inc}$ into the electric field integral equation. The electric field integral equation is [4, Eq. (12)]

$$-\frac{1}{\eta}\underline{E}_{tan}(\underline{J}) = \frac{1}{\eta}\underline{E}^{inc}_{tan} \qquad \text{on } S \tag{2}$$

where the subscript tan denotes the component tangent to S. In (2), $\eta$ is the impedance of free space, and $\underline{E}(\underline{J})$ is the electric field due to $\underline{J}$ in free space. This field is given by [16, Sec. 2.1]

$$\underline{E}(\underline{J}) = -j\eta[k \iint_S \underline{J}(\underline{r}')G(\underline{r}-\underline{r}')ds' + \frac{1}{k}\nabla \iint_S (\nabla'_s \cdot \underline{J}(\underline{r}'))G(\underline{r}-\underline{r}')ds'] \tag{3}$$

where

$$G(\underline{r}-\underline{r}') = \frac{e^{-jk|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} \tag{4}$$

In (3), k is the wave number, ds' is the differential element of area at $\underline{r}'$ on S, $\nabla'_s \cdot$ is the surface divergence on S with respect to the coordinates of $\underline{r}'$, and $\underline{r}$ is the position vector of the point at which $\underline{E}(\underline{J})$ is evaluated. In (2) $\underline{E}^{inc}$ is the incident electric field. The incident electric field is the electric field that would exist if S were absent.

The Rayleigh series for $\underline{J}$ is

$$\underline{J} = \sum_{n=0}^{\infty} k^n \underline{J}^{(n)} \tag{5}$$

where $\underline{J}^{(n)}$ is independent of k. It is assumed that the surface S on which $\underline{J}$ resides is the union of Q surfaces $S_1$, $S_2$,... $S_Q$ which are disjoint from each other.

$$S = \bigcup_{q=1}^{Q} S_q \tag{6}$$

In (6), $\bigcup$ denotes union. The surface $S_q$ may be either open or closed. If $S_q$ is closed, then it has no edges. If $S_q$ is open, it has an edge called $C_q$. It is assumed that $C_q$ consists of $R_q$ closed contours $\{C_{qr}$, r = 1,2,... $R_q\}$.

$$C_q = \bigcup_{r=1}^{R_q} C_{qr} \tag{7}$$

Since no line charge can accumulate on $C_{qr}$, $\underline{J}$ must satisfy

$$\underline{J} \cdot \underline{u}_b = 0 \quad \text{on } C_{qr} \;, \quad \begin{cases} q = 1,2,\ldots Q \\ r = 1,2,\ldots R_q \end{cases} \tag{8}$$

where $\underline{u}_b$ is the unit vector tangent to $S_q$ and normal to $C_{qr}$. It follows from (5) and (8) that

$$\underline{J}^{(0)} \cdot \underline{u}_b = 0 \quad \text{on } C_{qr} \tag{9}$$

and

$$\underline{J}^{(1)} \cdot \underline{u}_b = 0 \quad \text{on } C_{qr} \tag{10}$$

The ranges of values of q and r in (9) and (10) are the same as in (8). Similar boundary conditions hold for $\underline{J}^{(2)}$, $\underline{J}^{(3)}$,... but will not be used.

The Rayleigh series for $\underline{E}^{inc}$ is

$$\underline{E}^{inc} = \sum_{n=0}^{\infty} k^n \underline{E}^{(n)} \tag{11}$$

where $\underline{E}^{(n)}$ is independent of k. The expansion of $G(\underline{r}-\underline{r}')$ in powers of k is

$$G(\underline{r}-\underline{r}') = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(-jk)^n |\underline{r}-\underline{r}'|^{n-1}}{n!} \tag{12}$$

In view of (5) and (12), substitution of (3) and (11) into (2) yields

$$\frac{j}{4\pi} \sum_{n=1}^{\infty} k^n \sum_{m=0}^{n-1} \frac{(-j)^{n-m-1}}{(n-m-1)!} \left[ \iint_S |\underline{r}-\underline{r}'|^{n-m-2} \, \underline{J}^{(m)}(\underline{r}')ds' \right]_{tan}$$

$$+ \frac{j}{4\pi} \sum_{n=-1}^{\infty} k^n \sum_{m=0}^{n+1} \frac{(-j)^{n-m+1}}{(n-m+1)!} \nabla_s \iint_S |\underline{r}-\underline{r}'|^{n-m} \, \nabla'_s \cdot \underline{J}^{(m)}(\underline{r}')ds' = \frac{1}{\eta} \sum_{n=0}^{\infty} k^n \underline{E}^{(n)}_{tan}(\underline{r}) \text{ on } S \tag{13}$$

where $\nabla_s$ is the component of $\nabla$ tangent to S. The operator $\nabla_s$ is called the surface gradient on S.

Equation (13) implies that the coefficient of $k^n$ on the left-hand side of (13) is equal to the coefficient of $k^n$ on the right-hand side of (13) for n = -1, 0, 1, 2,... . Setting to zero the coefficient of 1/k on the left-hand side of (13), we obtain

$$\nabla_s \iint_S \frac{\nabla'_s \cdot \underline{J}^{(0)}(\underline{r}')}{|\underline{r}-\underline{r}'|} ds' = 0 \quad \text{on } S \tag{14}$$

In view of (6), (14) implies that

$$\iint_S \frac{\nabla'_s \cdot \underline{J}^{(0)}(\underline{r}')}{|\underline{r}-\underline{r}'|} ds' = C^{(0q)} \quad \text{on } S_q, \; q=1,2,\ldots Q \tag{15}$$

where $\{C^{(0q)}\}$ are unknown constants. We set $S = S_q$, $\phi = 1$, and $\underline{W} = \underline{J}^{(0)}$ in (A-1) and take advantage of (9) to obtain

$$\iint_{S_q} \nabla'_s \cdot \underline{J}^{(0)}(\underline{r}')ds' = 0, \quad q = 1, 2, \ldots Q \tag{16}$$

The electric charge associated with $\underline{J}^{(0)}(\underline{r}')$ by means of (1) is called $q^{(0)}(\underline{r}')$.

$$q^{(0)}(\underline{r}') = \frac{\nabla_s' \cdot \underline{J}^{(0)}(\underline{r}')}{-j\omega} \tag{17}$$

Now, (15) states that each of the surfaces $\{S_q\}$ is charged to a constant potential whereas (16) states that there is no net charge on any of the surfaces $\{S_q\}$. In this case, it is well-known from potential theory that $q^{(0)}(\underline{r}')$ must vanish everywhere on S. As a result, the magnetostatic current $\underline{J}^{(0)}$ satisfies

$$\nabla_s' \cdot \underline{J}^{(0)}(\underline{r}') = 0 \quad \text{on } S \tag{18}$$

Setting the constant with respect to k on one side of (13) equal to the constant on the other side of (13), we obtain

$$\frac{1}{4\pi\varepsilon_o} \nabla_s \iint_S \frac{q^{(1)}(\underline{r}')}{|\underline{r}-\underline{r}'|} ds' = \underline{E}_{tan}^{(0)}(\underline{r}) \quad \text{on } S \tag{19}$$

where $\varepsilon_o$ is the permittivity of free space and $q^{(1)}$ is the electric charge associated with $k\underline{J}^{(1)}$ by means of (1).

$$q^{(1)}(\underline{r}') = \frac{\nabla_s' \cdot \underline{J}^{(1)}(\underline{r}')}{-jc} \tag{20}$$

In (20), $c = 1/(\eta\varepsilon_o)$ is the speed of light. If

$$\nabla \times \underline{E}^{(0)} = 0 \tag{21}$$

then there is a scalar function $\phi^{(1)}(r)$ such that

$$\underline{E}^{(0)}(\underline{r}) = -\nabla\phi^{(1)}(\underline{r}) \tag{22}$$

and, with the help of (6), (19) will reduce to

$$\frac{1}{4\pi\varepsilon_o} \iint_S \frac{q^{(1)}(\underline{r}')}{|\underline{r}-\underline{r}'|} ds' = -\phi^{(1)}(\underline{r}) - C^{(1q)} \quad \text{on } S_q,\ q=1,2,\ldots Q \tag{23}$$

where $\{C^{(1q)}\}$ are unknown constants.

The following reasoning establishes (21). Substitution of the Rayleigh series (11) for $\underline{E}^{inc}$ and the Rayleigh series

$$\underline{H}^{inc} = \sum_{n=0}^{\infty} k^n \underline{H}^{(n)} \tag{24}$$

for the incident magnetic field $\underline{H}^{inc}$ into the Maxwell equation

$$\nabla \times \underline{E}^{inc} = - jk\eta \, \underline{H}^{inc} \tag{25}$$

gives

$$\sum_{n=0}^{\infty} k^n \nabla \times \underline{E}^{(n)} = - j\eta \sum_{n=1}^{\infty} k^n \underline{H}^{(n-1)} \tag{26}$$

Equation (21) is a consequence of (26).

We set $S = S_q$, $\phi = 1$, and $\underline{W} = \underline{J}^{(1)}$ in (A-1) and take advantage of (10) to obtain

$$\iint_{S_q} \nabla'_s \cdot \underline{J}^{(1)}(\underline{r}')ds' = 0 \ , \quad q = 1,2,\ldots Q \tag{27}$$

In view of (20), (27) implies that

$$\iint_{S_q} q^{(1)}(\underline{r}')ds' = 0 \ , \quad q = 1,2,\ldots \ Q \tag{28}$$

Now, the auxiliary equation (28) atones for the unknown constants $\{C^{(1q)}\}$ in (23) so that the pair of equations (23) and (28) suffices to determine $q^{(1)}$. It is evident that $q^{(1)}$ is the electrostatic charge. Accordingly, (23) and (28) a.e called the electrostatic equations.

Intending to determine the magnetostatic current $\underline{J}^{(0)}$, we let $\underline{W}$ be a differentiable vector function tangent to $\{S_q\}$ and integrate over $S_q$ the dot product of $\underline{W}$ with the terms proportional to k on both sides of (13) to obtain

$$j \iint_{S_q} \underline{W}(\underline{r}) \cdot \underline{A}^{(0)}(\underline{r})ds + \iint_{S_q} \underline{W}(\underline{r}) \cdot \nabla\phi(\underline{r})ds = \frac{1}{\eta} \iint_{S_q} \underline{W}(\underline{r}) \cdot \underline{E}^{(1)}(\underline{r})ds, \quad q=1,2,\ldots Q \tag{29}$$

In (29),

$$\underline{A}^{(0)}(\underline{r}) = \iint_{S} \frac{\underline{J}^{(0)}(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|} ds' \tag{30}$$

and, thanks to (18),

$$\phi = \frac{j}{4\pi} \iint_{S} \frac{\nabla_s' \cdot \underline{J}^{(2)}(\underline{r}')}{|\underline{r}-\underline{r}'|} ds' \tag{31}$$

Equation (29) will reduce to an equation for $\underline{J}^{(0)}$ alone if $\underline{W}(\underline{r})$ is chosen such that

$$\iint_{S_q} \underline{W}(\underline{r}) \cdot \nabla\phi(\underline{r})ds = 0, \quad q=1,2,\ldots Q \tag{32}$$

According to (A-1), (32) will be satisfied if

$$\nabla_s \cdot \underline{W} = 0 \quad \text{on } S_q, \qquad q=1,2,\ldots Q \tag{33}$$

and

$$\underline{W} \cdot \underline{u}_b = 0 \quad \text{on } C_{qr}, \quad \begin{cases} q=1,2,\ldots Q \\ r=1,2,\ldots R_q \end{cases} \tag{34}$$

If (33) is true, then, according to (B-1), there is a scalar function $u(\underline{r})$ such that

$$\underline{W}(\underline{r}) = \underline{n} \times \nabla_s u(\underline{r}) \quad \text{on } S_q, \; q=1,2,\ldots Q \tag{35}$$

where $\underline{n}$ is the unit vector normal to S. If (34) is also true, then $u(\underline{r})$ must satisfy

$$u(\underline{r}) = U_{qr} \text{ on } C_{qr}, \quad \begin{cases} q=1,2,\ldots Q \\ r=1,2,\ldots R_q \end{cases} \tag{36}$$

where $\{U_{qr}\}$ are unknown constants.

In view of (32), substitution of (35) into the right-hand side of (29) yields

$$j \iint_{S_q} \underline{W}(\underline{r}) \cdot \underline{A}^{(0)}(\underline{r})ds = \frac{1}{\eta} \iint_{S_q} (\underline{n} \times \nabla_s u(\underline{r})) \cdot \underline{E}^{(1)}(\underline{r})ds, \quad q=1,2,\ldots Q \quad (37)$$

Application of (C-1) to the integral on the right-hand side of (37) and subsequent use of (36) give

$$j \iint_{S_q} \underline{W} \cdot \underline{A}^{(0)}ds = -\frac{1}{\eta} \iint_{S_q} u(\nabla \times \underline{E}^{(1)}) \cdot \underline{n}\, ds$$

$$+ \frac{1}{\eta} \sum_{r=1}^{R_q} U_{qr} \int_{C_{qr}} \underline{E}^{(1)} \cdot \underline{u}_\ell d\ell \;, \quad q=1,2,\ldots Q \quad (38)$$

In (38), $\underline{u}_\ell$ is the unit vector tangent to the contour $C_{qr}$. A right-handed screw would advance in the direction of $\underline{n}$ when turned in the direction of $\underline{u}_\ell$.

It is evident from (26) that

$$\nabla \times \underline{E}^{(1)} = -j\eta\, \underline{H}^{(0)} \quad (39)$$

Application of Stokes' theorem [17, Eq. (42) on p. 489] to the integral over $C_{qr}$ in (38), subsequent use of (39), and division of both sides of (38) by j give

$$\iint_{S_q} \underline{W} \cdot \underline{A}^{(0)}ds = \iint_{S_q} u\, \underline{H}^{(0)} \cdot \underline{n}\, ds - \sum_{r=1}^{R_q} U_{qr} \iint_{S_{qr}} \underline{H}^{(0)} \cdot \underline{n}\, ds, \quad q=1,2,\ldots Q \quad (40)$$

In (40), $S_{qr}$ is a cap surface over the contour $C_{qr}$. On $S_{qr}$, $\underline{n}$ is the unit vector normal to $S_{qr}$. The direction of $\underline{n}$ on $S_{qr}$ is related to the direction of $\underline{u}_\ell$ in (38) by the right-hand rule stated prior to (39).

Equations (35), (C-1), and (36) transform the left-hand side of (40) to

$$-\iint_{S_q} u\,\underline{n}\cdot\nabla\times\underline{A}^{(0)}ds + \sum_{r=1}^{R_q} U_{qr}\oint_{C_{qr}} \underline{A}^{(0)}\cdot\underline{u}_\ell d\ell$$

Stokes' theorem [17, Eq. (42) on p. 489] transforms the above expression to

$$-\iint_{S_q} u\,\underline{n}\cdot\nabla\times\underline{A}^{(0)}ds + \sum_{r=1}^{R_q} U_{qr}\iint_{S_{qr}} \underline{n}\cdot\nabla\times\underline{A}^{(0)}ds$$

Therefore, (40) becomes

$$\iint_{S_q} u(\underline{n}\cdot\nabla\times\underline{A}^{(0)} + \underline{n}\cdot\underline{H}^{(0)})ds = \sum_{r=1}^{R_q} U_{qr}\iint_{S_{qr}} (\underline{n}\cdot\nabla\times\underline{A}^{(0)} + \underline{n}\cdot\underline{H}^{(0)})ds, \quad q=1,2,\ldots Q \tag{41}$$

In (41), u is any differentiable scalar function that reduces to the arbitrary constant $U_{qr}$ on the contour $C_{qr}$ for $r = 1,2,\ldots R_q$. Now, (41) can be valid for all such functions u only if

$$-\underline{n}\cdot\nabla\times\underline{A}^{(0)} = \underline{n}\cdot\underline{H}^{(0)} \quad \text{on} \quad S_q,\ q=1,2,\ldots Q \tag{42}$$

$$-\iint_{S_{qr}} \underline{n}\cdot\nabla\times\underline{A}^{(0)}ds = \iint_{S_{qr}} \underline{n}\cdot\underline{H}^{(0)}ds, \quad \begin{cases} q=1,2,\ldots Q \\ r=1,2,\ldots R_q \end{cases} \tag{43}$$

Since $\underline{A}^{(0)}$ is given by (30), the pair of equations (42) and (43) helps to determine $\underline{J}^{(0)}$. Equations (9) and (18) state that there is no electric change associated with $\underline{J}^{(0)}$. We believe that (9), (18), (42), and (43) uniquely determine the magnetostatic current $\underline{J}^{(0)}$. Accordingly, these equations are called the magnetostatic equations.

Equation (9) is true because no line charge can exist. Equation (18) is similar to [11, Eq. (1.80)]. Equation (42) is a statement of the well-known fact that the normal component of the total magnetic field is zero on a conducting surface [11, Eq. (1.90)]. Equation (43) can be more directly obtained in the following manner. The line integrals over $C_{qr}$ of the terms proportional to k on both sides of (13) are

$$j \int_{C_{qr}} \underline{A}^{(0)} \cdot \underline{u}_{\ell}\, d\ell + \int_{C_{qr}} \nabla\phi \cdot \underline{u}_{\ell}\, d\ell = \frac{1}{\eta} \int_{C_{qr}} \underline{E}^{(1)} \cdot \underline{u}_{\ell}\, d\ell \quad (44)$$

where $\underline{A}^{(0)}$ and $\phi$ are given by (30) and (31), respectively. Since $C_{qr}$ is a closed loop, the second integral on the left-hand side of (44) vanishes. Application of Stokes' theorem [17, Eq. (42) on p. 489] to the remaining integrals in (44), subsequent use of (39), and multiplication by j give (43).

Equation (43) is not commonplace because, as will be shown, (43) is necessary only if $S_q$ is open and is bounded by two or more closed contours $\{C_{qr}\}$. If $S_q$ is not open, then it is closed and therefore not bounded by any contour. Consequently, there are no surfaces $S_{qr}$ so that (43) is absent. If $S_q$ is open and is bounded by one closed contour $C_{q1}$, then $S_{q1}$ is identical to $S_q$. As a result, (43) is redundant because it can be obtained by integrating (42) over $S_q$. If $S_q$ is open and is bounded by $R_q$ closed contours $\{C_{qr}, r=1,2,\ldots R_q\}$ where $R_q \geq 2$, then

$$S_q = \bigcup_{r=1}^{R_q} S_{qr} \quad (45)$$

As a result, the integral of (42) over $S_q$ is the sum of equations (43) for $r=1,2,\ldots R_q$. In this case, (43) must be enforced for $R_q-1$ values of r. At first glance, there appears to be more surface area on the right-hand side of (45) than on the left-hand side. However, portions of surface area with oppositely directed normal vectors cancel each other on the right-hand side of (45) so that (45) is true.

## III. CONSTRUCTION OF THE NEW E-FIELD SOLUTION

The new E-field solution is a moment solution to the electric field integral equation (2). The new E-field solution is constructed by expanding the electric current $\underline{J}$ as

$$\underline{J} = \sum_{j=1}^{N_m} I_j^m \underline{J}_j^m + \sum_{j=1}^{N_e} I_j^e k\rho_j \underline{J}_j^e \tag{46}$$

where $\underline{J}_j^m$ and $\underline{J}_j^e$ are vector functions that are on S and are tangent to S. According to (6), S consists of the surfaces $\{S_q\}$. For convenience, $\underline{J}_j^m$ is chosen to be non-trivial only on $S_q$ for q equal to the single value m(j). Similarly, $\underline{J}_j^e$ is chosen to be non-trivial only on $S_{e(j)}$. Furthermore, $\{\underline{J}_j^m\}$ is a suitable basis for expanding the magnetostatic current $\underline{J}^{(0)}$ and $\{\nabla_s \cdot \underline{J}_j^e\}$ is a suitable basis for expanding the electrostatic charge $q^{(1)}$. Neither $\underline{J}_j^m$ nor $\underline{J}_j^e$ is allowed to depend on k. The magnitudes of $\underline{J}_j^m$ and $\underline{J}_j^e$ should be comparable with each other. Because of (8), $\underline{J}_j^m$ must satisfy

$$\underline{J}_j^m \cdot \underline{u}_b = 0 \quad \text{on } C_{qr}\,, \quad \begin{cases} j=1,2,\ldots N_m \\ q=m(j) \\ r=1,2,\ldots R_{m(j)} \end{cases} \tag{47}$$

and $\underline{J}_j^e$ must satisfy

$$\underline{J}_j^e \cdot \underline{u}_b = 0 \quad \text{on } C_{qr}\,, \quad \begin{cases} j=1,2,\ldots N_e \\ q=e(j) \\ r=1,2,\ldots R_{e(j)} \end{cases} \tag{48}$$

In view of (18), $\underline{J}_j^m$ must also satisfy

$$\nabla_s' \cdot \underline{J}_j^m(\underline{r}') = 0 \quad \text{on } S_{m(j)},\ j=1,2,\ldots N_m \tag{49}$$

In (46), $I_j^m$ and $I_j^e$ are unknown coefficients to be determined. In general, these coefficients will depend on k. The scale factors $\{k\rho_j\}$ in (46) are for later convenience. Here, $\rho_j$ is a length so that $k\rho_j$ is dimensionless. The exact value of $\rho_j$ is not critical. However, the order of magnitude of $\rho_j$ should be that of a dimension of S. It is evident from (46) that the expansion functions for $\underline{J}$ are

$$\{\underline{J}_j^m \ , \ j=1,2,\ldots N_m\} \tag{50a}$$

and

$$\{k\rho_j\underline{J}_j^e \ , \ j=1,2,\ldots N_e\} \tag{50b}$$

In analogy with the above expansion functions, testing functions

$$\{\underline{W}_i^m \ , \ i=1,2,\ldots N_m\} \tag{51a}$$

and

$$\{k\rho_i\underline{W}_i^e \ , \ i=1,2,\ldots N_e\} \tag{51b}$$

are introduced on S. Both $\underline{W}_i^m$ and $\underline{W}_i^e$ are tangent to S. $\underline{W}_i^m$ is chosen to be non-trivial only on $S_{m(i)}$. $\underline{W}_i^e$ is chosen to be non-trivial only on $S_{e(i)}$. Neither $\underline{W}_i^m$ nor $\underline{W}_i^e$ is allowed to depend on k. The magnitudes of $\underline{W}_i^m$ and $\underline{W}_i^e$ should be comparable with each other.

The testing function $\underline{W}_i^m$ is chosen to satisfy

$$\underline{W}_i^m \cdot \underline{u}_b = 0 \quad \text{on } C_{m(i),r}, \ i=1,2,\ldots N_m \tag{52}$$

and $\underline{W}_i^e$ is chosen to satisfy

$$\underline{W}_i^e \cdot \underline{u}_b = 0 \quad \text{on } C_{e(i),r}, \ i=1,2,\ldots N_e \tag{53}$$

Moreover, $\{\nabla_s \cdot \underline{W}_i^e\}$ should be a suitable set of testing functions for the electrostatic equation (23). Unfortunately, $\{\underline{W}_i^m\}$ can not be a suitable set of testing functions for the magnetostatic equation (42) because (42) is a scalar equation. However, taking a cue from (49), we require that

$$\nabla_s \cdot \underline{W}_i^m(\underline{r}) = 0 \quad \text{on} \quad S_{m(i)},\ i=1,2,\ldots N_m \tag{54}$$

where $\nabla_s \cdot$ is the surface divergence with respect to the coordinate of $\underline{r}$. Thanks to (54) and (B-1), there are scalar functions $\{u_i\}$ such that

$$\underline{W}_i^m(\underline{r}) = \underline{n} \times \nabla_s u_i(\underline{r}) \ , \quad i = 1,\ 2,\ldots N_m \tag{55}$$

Furthermore, $\{\underline{W}_i^m\}$ are chosen such that if an expansion for $\underline{J}^{(0)}$ in terms of $\{\underline{J}_j^m\}$ is entered into (40) by means of (30), then enforcement of (40) for $\underline{W}$ successively equal to each member of $\{\underline{W}_i^m\}$ determines the coefficients in the expansion for $\underline{J}^{(0)}$.

The symmetric product of two vector functions is defined to be the integral of their dot product over S. If (46) is substituted into (3) and if (3) is subsequently substituted into (2), then the symmetric products of (2) with each of the testing functions (51) form the matrix equation

$$\begin{bmatrix} Z^{mm} & Z^{me} \\ Z^{em} & Z^{ee} \end{bmatrix} \begin{bmatrix} \vec{I}^m \\ \vec{I}^e \end{bmatrix} = \begin{bmatrix} \vec{V}^m \\ \vec{V}^e \end{bmatrix} \tag{56}$$

In (56), $\vec{I}^m$ is the column vector of the coefficients $\{I_j^m\}$ in (46), and $\vec{I}^e$ is the column vector of the coefficients $\{I_j^e\}$ in (46). Also, $Z^{mm}$, $Z^{em}$, $Z^{me}$, and $Z^{ee}$ are submatrices whose ijth elements are given by

$$Z_{ij}^{mm} = jk \iint_{S_{m(i)}} ds\ \underline{W}_i^m(\underline{r}) \cdot \iint_{S_{m(j)}} ds'\ \underline{J}_j^m(\underline{r}')G(\underline{r}-\underline{r}') \tag{57}$$

$$Z_{ij}^{em} = jk^2\rho_i \iint_{S_{e(i)}} ds\ \underline{W}_i^e(\underline{r}) \iint_{S_{m(j)}} ds'\ \underline{J}_j^m(\underline{r}')G(\underline{r}-\underline{r}') \tag{58}$$

$$Z_{ij}^{me} = jk^2\rho_j \iint_{S_{m(i)}} ds\, \underline{W}_i^m(\underline{r}) \cdot \iint_{S_{e(j)}} ds'\, \underline{J}_j^e(\underline{r}')G(\underline{r}-\underline{r}')$$

$$+ j\rho_j \iint_{S_{m(i)}} ds\, \underline{W}_i^m(\underline{r}) \cdot \nabla \iint_{S_{e(j)}} ds'(\nabla_s' \cdot \underline{J}_j^e(\underline{r}'))G(\underline{r}-\underline{r}') \tag{59}$$

$$Z_{ij}^{ee} = jk^3\rho_i\rho_j \iint_{S_{e(i)}} ds\, \underline{W}_i^e(\underline{r}) \cdot \iint_{S_{e(j)}} ds'\, \underline{J}_j^e(\underline{r}')G(\underline{r}-\underline{r}')$$

$$+ jk\rho_i\rho_j \iint_{S_{e(i)}} ds\, \underline{W}_i^e(\underline{r}) \cdot \nabla \iint_{S_{e(j)}} ds'(\nabla_s' \cdot \underline{J}_j^e(\underline{r}'))G(\underline{r}-\underline{r}') \tag{60}$$

Moreover, $\vec{V}^m$ and $\vec{V}^e$ are column vectors whose ith elements are given by

$$V_i^m = \frac{1}{\eta} \iint_{S_{m(i)}} \underline{W}_i^m \cdot \underline{E}^{inc} ds \tag{61}$$

$$V_i^e = \frac{k\rho_i}{\eta} \iint_{S_{e(i)}} \underline{W}_i^e \cdot \underline{E}^{inc} ds \tag{62}$$

In the transition from (2) to (57) - (62), it was permissible to omit the subscript tan because $\underline{W}_i^m$ and $\underline{W}_i^e$ are tangent to S.

Thanks to (A-1), (52), and (54), (59) reduces to

$$Z_{ij}^{me} = jk^2\rho_j \iint_{S_{m(i)}} ds\, \underline{W}_i^m(\underline{r}) \cdot \iint_{S_{e(j)}} ds'\, \underline{J}_j^e(\underline{r}')G(\underline{r}-\underline{r}') \tag{63}$$

In view of (53), application of (A-1) to (60) yields

$$Z_{ij}^{ee} = jk^3\rho_i\rho_j \iint_{S_{e(i)}} ds\, \underline{W}_i^e(\underline{r}) \cdot \iint_{S_{e(j)}} ds'\, \underline{J}_j^e(\underline{r}')G(\underline{r}-\underline{r}')$$

$$- jk\rho_i\rho_j \iint_{S_{e(i)}} ds(\nabla_s \cdot \underline{W}_i^e(\underline{r})) \iint_{S_{e(j)}} ds'(\nabla_s' \cdot \underline{J}_j^e(\underline{r}'))G(\underline{r}-\underline{r}') \tag{64}$$

Substitution of (55) into (61) and subsequent application of (C-1) give

$$V_i^m = -\frac{1}{\eta} \iint\limits_{S_{m(i)}} u_i(\nabla \times \underline{E}^{inc}) \cdot \underline{n}\, ds + \frac{1}{\eta} \sum_{r=1}^{R_{m(i)}} \int\limits_{C_{m(i),r}} u_i(\underline{E}^{inc} \cdot \underline{u}_\ell) d\ell \tag{65}$$

It follows from (52) and (55) that $u_i$ is constant on $C_{m(i),r}$. Hence, (65) becomes

$$V_i^m = -\frac{1}{\eta} \iint\limits_{S_{m(i)}} u_i(\nabla \times \underline{E}^{inc}) \cdot \underline{n}\, ds + \frac{1}{\eta} \sum_{r=1}^{R_{m(i)}} K_{ir} \int\limits_{C_{m(i),r}} \underline{E}^{inc} \cdot \underline{u}_\ell d\ell \tag{66}$$

where $K_{ir}$ is the value of $u_i$ on $C_{m(i),r}$. Application of Stokes' theorem to the line integral in (66) gives

$$V_i^m = -\frac{1}{\eta} \iint\limits_{S_{m(i)}} u_i(\nabla \times \underline{E}^{inc}) \cdot \underline{n}\, ds + \frac{1}{\eta} \sum_{r=1}^{R_{m(i)}} K_{ir} \iint\limits_{S_{m(i),r}} \underline{n} \cdot \nabla \times \underline{E}^{inc} ds \tag{67}$$

Equation (25) reduces (67) to

$$V_i^m = jk \iint\limits_{S_{m(i)}} u_i(\underline{H}^{inc} \cdot \underline{n}) ds - jk \sum_{r=1}^{R_{m(i)}} K_{ir} \iint\limits_{S_{m(i),r}} \underline{H}^{inc} \cdot \underline{n}\, ds \tag{68}$$

It is evident from (55) and (61) that addition of a constant to $u_i$ should not affect the value of $V_i^m$. Addition of a constant to $u_i$ does not affect the value (68) of $V_i^m$ because, according to (45),

$$S_{m(i)} = \bigcup_{r=1}^{R_{m(i)}} S_{m(i),r} \tag{69}$$

If some linear combination of the $\{u_i\}$ were equal to a constant, then the corresponding linear combination of the $\{\underline{W}_i^m\}$ of (55) would be zero. In this case, the matrix of the superscripted Z's on the left-hand side of (56) would be singular. To avoid a singular matrix, the $\{u_i\}$ should be chosen so that no linear combination of them is equal to a constant.

Construction of the new E-field solution is now complete. The new E-field solution for $\underline{J}$ is given by (46) where the coefficients $\{I_j^m\}$ and $\{I_j^e\}$ are the elements of the column vectors $\vec{I}^m$ and $\vec{I}^e$ that satisfy (56). The ijth elements of the submatrices $Z^{mm}$, $Z^{em}$, $Z^{me}$, and $Z^{ee}$ in (56) are given by (57), (58), (63), and (64), respectively. The ith elements of the column vectors $\vec{V}^m$ and $\vec{V}^e$ in (56) are given by (68) and (62), respectively.

## IV. LOW FREQUENCY BEHAVIOR OF THE NEW E-FIELD SOLUTION

If the wave number k is sufficiently small, the elements of $Z^{mm}$, $Z^{ee}$, $\vec{V}^m$, and $\vec{V}^e$ in (56) are proportional to k whereas the elements of $Z^{em}$ and $Z^{me}$ are proportional to $k^2$. As a result, (56) is well approximated by the pair of equations

$$Z^{mmo}\vec{I}^m = \vec{V}^{mo} \tag{70}$$

$$Z^{eeo}\vec{I}^e = \vec{V}^{eo} \tag{71}$$

whenever k is sufficiently small. Here, $Z^{mmo}$, $\vec{V}^{mo}$, $Z^{eeo}$, and $\vec{V}^{eo}$ are the low frequency limits of $Z^{mm}$, $\vec{V}^m$, $Z^{ee}$, and $\vec{V}^e$, respectively. From (57), (68), (64), and (62), we obtain

$$Z_{ij}^{mmo} = jk \iint_{S_{m(i)}} ds\, \underline{W}_i^m(\underline{r}) \cdot \iint_{S_{m(j)}} ds' \frac{\underline{J}_j^m(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|} \tag{72}$$

$$V_i^{mo} = jk \iint_{S_{m(i)}} u_i(\underline{H}^{(0)} \cdot \underline{n})ds - jk \sum_{r=1}^{R_{m(i)}} K_{ir} \iint_{S_{m(i),r}} \underline{H}^{(0)} \cdot \underline{n}\, ds \tag{73}$$

$$Z_{ij}^{eeo} = -jk\rho_i\rho_j \iint_{S_{e(i)}} ds(\nabla_s \cdot \underline{W}_i^e(\underline{r})) \iint_{S_{e(j)}} ds' \frac{\nabla_s' \cdot \underline{J}_j^e(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|} \tag{74}$$

$$V_i^{eo} = \frac{k\rho_i}{\eta} \iint_{S_{e(i)}} \underline{W}_i^e \cdot \underline{E}^{(0)} ds \tag{75}$$

where $\underline{E}^{(0)}$ and $\underline{H}^{(0)}$ have been extracted from the right-hand sides of (11) and (24), respectively.

Later in this Section, it is shown that (70) is the matrix equation that appears in a moment solution for the magnetostatic current $\underline{J}^{(0)}$. It is also shown that (71) is the matrix equation that appears in a moment solution for the electrostatic charge $q^{(1)}$. Presumably, the matrix equations for the magnetostatic current and the electrostatic charge can be solved easily. If this is true, then the matrix equations (70) and (71) can be solved easily. Hence, the matrix equation (56) can be solved easily when the frequency is low, and the solution will tend to give the magnetostatic current and the electrostatic charge.

A moment solution for the magnetostatic current $\underline{J}^{(0)}$ is now constructed. The matrix equation that appears in this solution will be (70). In view of (30), (40) is an equation for $\underline{J}^{(0)}$. Upon substitution of $jk\underline{W}_i^m$ for $\underline{W}$ and m(i) for q, (40) becomes

$$jk \iint_{S_{m(i)}} \underline{W}_i^m \cdot \underline{A}^{(0)} ds = jk \iint_{S_{m(i)}} u_i \underline{H}^{(0)} \cdot \underline{n}\, ds$$

$$- jk \sum_{r=1}^{R_{m(i)}} K_{ir} \iint_{S_{m(i),r}} \underline{H}^{(0)} \cdot \underline{n}\, ds\ , \quad i=1,2,\ldots N_m \tag{76}$$

where $u_i$ is a scalar function that satisfies (55) and $K_{ir}$ is the value of $u_i$ on $C_{m(i),r}$. If the expansion

$$\underline{J}^{(0)} = \sum_{j=1}^{N_m} I_j^{mo} \underline{J}_j^m \tag{77}$$

is inserted into (30) and if (30) is substituted into (76), then equations (76) will form the matrix equation

$$Z^{mmo} \vec{I}^{mo} = \vec{V}^{mo} \tag{78}$$

Here, $\vec{I}^{mo}$ is the column vector of coefficients $\{I_j^{mo}\}$ in (77). In (78), the elements of $Z^{mmo}$ and $\vec{V}^{mo}$ are given by (72) and (73), respectively. As expected, the matrix equation (78) for $\vec{I}^{mo}$ is the same as the matrix equation (70) for $\vec{I}^m$.

A moment solution for the electrostatic charge $q^{(1)}$ is now constructed. The matrix equation that appears in this solution will be (71). Substitution of the expansion

$$q^{(1)}(\underline{r}') = \frac{j}{c} \sum_{j=1}^{N_e} I_j^{eo} \rho_j (\nabla_s' \cdot \underline{J}_j^e(\underline{r}')) \tag{79}$$

into the electrostatic equation (23) and integration of the product of (23) with $-(k\rho_i/\eta)\nabla_s \cdot \underline{W}_i^e$ over $S_{e(i)}$ produces

$$-jk\rho_i \sum_{j=1}^{N_e} I_j^{eo} \rho_j \iint_{S_{e(i)}} ds (\nabla_s \cdot \underline{W}_i^e(\underline{r})) \iint_{S_{e(j)}} ds' \frac{\nabla_s' \cdot \underline{J}_j^e(\underline{r}')}{4\pi |\underline{r}-\underline{r}'|}$$

$$= \frac{k\rho_i}{\eta} \iint_{S_{e(i)}} \phi^{(1)}(\underline{r}) \nabla_s \cdot \underline{W}_i^e(\underline{r})\, ds$$

$$+ \frac{k\rho_i C^{(1,e(i))}}{\eta} \iint_{S_{e(i)}} \nabla_s \cdot \underline{W}_i^e(\underline{r}) ds, \quad i=1,2,\ldots N_e \tag{80}$$

Thanks to (53) and the divergence theorem [17, Eq. (42) on p. 503], the second integral on the right-hand side of (80) vanishes. Next, (A-1) is applied to the first integral on the right-hand side of (80), and then (22) is used to replace the resulting $\nabla\phi^{(1)}$ by $-\underline{E}^{(0)}$. In view of these considerations, equations (80) form the matrix equation

$$Z^{eeo}\,\vec{I}^{eo} = \vec{V}^{eo} \tag{81}$$

Here, $\vec{I}^{eo}$ is the column vector of the coefficients $\{I_j^{eo}\}$ in (79). In (81), the elements of $Z^{eeo}$ and $\vec{V}^{eo}$ are given by (74) and (75), respectively. As expected, the matrix equation (81) for $\vec{I}^{eo}$ is the same as the matrix equation (71) for $\vec{I}^{e}$.

## V. APPLICATION TO A SURFACE OF REVOLUTION

If the surface S is a surface of revolution, suitable expansion functions $\{\underline{J}_j^m\}$ and $\{k\rho_j\underline{J}_j^e\}$ for the new E-field solution can be constructed by taking linear combinations of the expansion functions $\{\underline{J}_{nj}^t\}$ and $\{\underline{J}_{nj}^\phi\}$ defined by [5, Eqs. (2) and (3)]

$$\underline{J}_{nj}^t = \underline{u}_t \frac{T_j(t)}{\rho} e^{jn\phi}\,, \quad \begin{cases} j=1,2,\ldots P-2 \\ n=0,\pm1,\pm2,\ldots \end{cases} \tag{82}$$

$$\underline{J}_{nj}^\phi = \underline{u}_\phi \frac{P_j(t)}{\rho_j} e^{jn\phi}\,, \quad \begin{cases} j=1,2,\ldots P-1 \\ n=0,\pm1,\pm2,\ldots \end{cases} \tag{83}$$

Here, t is the arc length along the generating curve of S, and $\phi$ is the azimuthal angle. t and $\phi$ are orthogonal coordinates on S. $\underline{u}_t$ and $\underline{u}_\phi$ are unit vectors in the t and $\phi$ directions, respectively. Assuming that $\bar{t}_1, \bar{t}_2 \ldots \bar{t}_P$ are points on the generating curve, $T_j(t)$ is the triangle function which begins at $t_j^-$, peaks at $t_{j+1}^-$, and ends at $t_{j+2}^-$. $\rho$ is the distance from the axis about which the generating curve is rotated. $P_j(t)$ is the unit pulse function whose domain extends from $t_j^-$ to $t_{j+1}^-$. $\rho_j$ is the value of $\rho$ at the center of the domain of $P_j(t)$.

To indicate dependence on n, the expansion functions for the new E-field solution for a surface of revolution are called $\{\underline{J}_{nj}^m\}$ and $\{k\rho_j \underline{J}_{nj}^e\}$ instead of $\{\underline{J}_j^m\}$ and $\{k\rho_j \underline{J}_j^e\}$. Similarly, the testing functions for the new E-field solution are called $\{\underline{W}_{ni}^m\}$ and $\{k\rho_i \underline{W}_{ni}^e\}$ instead of $\{\underline{W}_i^m\}$ and $\{k\rho_i \underline{W}_i^e\}$. The superscript m stands for magnetostatic, and the superscript e stands for electrostatic. In [5], the testing functions are the complex conjugates of the expansion functions. Accordingly, the testing functions for the new E-field solution are chosen to be the complex conjugates of the expansion functions for the new E-field solution.

$$\underline{W}_{ni}^m = \underline{J}_{ni}^{m*} \tag{84}$$

$$k\rho_i \, \underline{W}_{ni}^e = k\rho_i \, \underline{J}_{ni}^{e*} \tag{85}$$

Here, * denotes complex conjugate.

Since $\underline{J}_{nj}^m$ and $k\rho_j \underline{J}_{nj}^e$ are linear combinations of (82) and (83), $\underline{J}_{nj}^m$ and $k\rho_j \underline{J}_{nj}^e$ are proportional to $e^{jn\phi}$. It can be shown that the field due to any electric current proportional to $e^{jn\phi}$ is also proportional to $e^{jn\phi}$. Hence, the symmetric products of the fields due to $\underline{J}_{nj}^m$ and $k\rho_j \underline{J}_{nj}^e$ with $\underline{W}_{pi}^m$ and $k\rho_i \underline{W}_{pi}^e$ are zero for all values of p except p=n. As a result, the matrix equation (56) disintegrates into many "smaller" matrix equations, one for each value of n in (82).

$$\begin{bmatrix} Z_n^{mm} & Z_n^{me} \\ Z_n^{em} & Z_n^{ee} \end{bmatrix} \begin{bmatrix} \vec{I}_n^m \\ \vec{I}_n^e \end{bmatrix} = \begin{bmatrix} \vec{V}_n^m \\ \vec{V}_n^e \end{bmatrix}, \quad n=0,\pm1,\pm2,\ldots \tag{86}$$

According to (49), the surface divergence of $\underline{J}^m_{nj}$ must vanish. This means that $\underline{J}^m_{nj}$ can not have any electric charge associated with it. In order to construct $\underline{J}^m_{nj}$ as a linear combination of the functions in (82) and (83), we have to know the electric charges associated with these functions. The surface density of charge associated with $\underline{J}^t_{nj}$ is called $q^t_{nj}$ and is given by the equation of continuity.

$$q^t_{nj} = \frac{1}{-j\omega} \nabla_s \cdot \underline{J}^t_{nj} \tag{87}$$

Since $\underline{J}^t_{nj}$ is given by (82) and $\nabla_s \cdot$ is given by (B-3), (87) becomes

$$q^t_{nj} = \frac{1}{-j\omega\rho} \left(\frac{P_j(t)}{\Delta_j} - \frac{P_{j+1}(t)}{\Delta_{j+1}}\right) e^{jn\phi} \tag{88}$$

where $\Delta_j$ is the distance from $t^-_j$ to $t^-_{j+1}$. The surface density $q^\phi_{nj}$ of electric charge associated with $\underline{J}^\phi_{nj}$ is given by

$$q^\phi_{nj} = \frac{1}{-j\omega} \nabla_s \cdot \underline{J}^\phi_{nj} \tag{89}$$

which becomes

$$q^\phi_{nj} = \frac{-n}{\omega\rho_j\rho} P_j(t) e^{jn\phi} \tag{90}$$

Noting that $\underline{J}^\phi_{0j}$ has no electric charge associated with it, we choose

$$\underline{J}^m_{0j} = \underline{J}^\phi_{0j} , \quad j=1,2,\ldots P-1 \tag{91}$$

and

$$k\rho_j \underline{J}^e_{0j} = \underline{J}^t_{0j} , \quad j=1,2,\ldots P-2 \tag{92}$$

From (84) and (85), the corresponding testing functions are

$$\underline{W}^m_{0i} = \underline{J}^\phi_{0i} , \quad i=1,2,\ldots P-1 \tag{93}$$

$$k\rho_i \underline{W}^e_{0i} = \underline{J}^t_{0i} , \quad i=1,2,\ldots P-2 \tag{94}$$

Thus, for n=0, the expansion and testing functions for the new E-field solution are the same as those used in [5].

Equations (92) and (82) imply that $\underline{J}_{0j}^{e}$ is proportional to 1/k. However, $\underline{J}_{nj}^{e}$ is not allowed to depend on k in Section III. If $\underline{J}_{nj}^{e}$ depended on k, it might be difficult to obtain an accurate numerical solution to (86) because (86) would not be properly scaled. On the other hand, $Z_0^{em}$ and $Z_0^{me}$ are exactly zero so that, for n=0, (86) separates into two matrix equations, one involving $Z_0^{mm}$, the other involving $Z_0^{ee}$. In this case, it does not matter how $Z_0^{ee}$ is scaled with respect to $Z_0^{mm}$.

In order to calculate $\vec{V}_0^m$ from (68), a scalar function $u_{0i}$ must be found such that

$$\underline{W}_{0i}^{m} = \underline{n} \times \nabla_s u_{0i} \tag{95}$$

If

$$\underline{n} = \underline{u}_{\phi} \times \underline{u}_t \tag{96}$$

then it is not difficult to show that (95) is satisfied by

$$u_{0i} = \begin{cases} 0 & t \le t_i^- \\ \dfrac{t_i^- - t}{\rho_i} & t_i^- \le t \le t_{i+1}^- \\ -\dfrac{\Delta_i}{\rho_i} & t \ge t_{i+1}^- \end{cases} \tag{97}$$

Equation (68) was derived to show that $V_i^m$ is proportional to k for small k. If $\underline{E}^{inc}$ is such that the integration on the right-hand side of (61) yields an expression that is explicitly proportional to k for small k, then (68) is not necessary. As is evident from [5, Eq. (80)], such is the case for n=0 and for an obliquely incident plane wave. Thus, for plane wave incidence, the matrix equation [5, Eq. (6)] is adequate for n=0.

If $n \neq 0$, it is evident from (88) and (90) that

$$q^{t}_{nj} + \alpha_{nj} q^{\phi}_{nj} - \alpha_{n,j+1} q^{\phi}_{n,j+1} = 0 \tag{98}$$

where

$$\alpha_{nj} = \frac{j\rho_j}{n\Delta_j} \tag{99}$$

Therefore, it is suitable to choose

$$\underline{J}^{m}_{nj} = \underline{J}^{t}_{nj} + \alpha_{nj} \underline{J}^{\phi}_{nj} - \alpha_{n,j+1} \underline{J}^{\phi}_{n,j+1} \; , \; \begin{cases} j=1,2,\ldots P-2 \\ n=\pm 1, \pm 2, \ldots \end{cases} \tag{100}$$

The $\{k\rho_j \underline{J}^{e}_{nj}\}$ are defined by

$$k\rho_j \underline{J}^{e}_{nj} = k\rho_j \underline{J}^{\phi}_{nj} \quad , \quad \begin{cases} j=1,2,\ldots P-1 \\ n=\pm 1, \pm 2, \ldots \end{cases} \tag{101}$$

From (84) and (85), the corresponding testing functions are

$$\underline{W}^{m}_{ni} = \underline{J}^{t*}_{ni} - \alpha_{ni} \underline{J}^{\phi *}_{ni} + \alpha_{n,i+1} \underline{J}^{\phi *}_{n,i+1} \quad , \quad \begin{cases} i=1,2,\ldots P-2 \\ n=\pm 1, \pm 2, \ldots \end{cases} \tag{102}$$

and

$$k\rho_i \underline{W}^{e}_{ni} = k\rho_i \underline{J}^{\phi *}_{ni} \; , \; \begin{cases} i=1,2,\ldots P-1 \\ n=\pm 1, \pm 2, \ldots \end{cases} \tag{103}$$

In view of (96), it is not difficult to show that

$$\underline{W}^{m}_{ni} = \underline{n} \times \nabla_s \, u_{ni} \tag{104}$$

where

$$u_{ni} = \frac{j}{n} T_i(t) \, e^{-jn\phi} \tag{105}$$

Since the expansion and testing functions for the new E-field solution are linear combinations of the expansion and testing functions used in [5], the elements of the superscripted $Z_n$'s in (86) are linear combinations of the elements of the superscripted $Z_n$'s in [5, Eq. (6)]. Of course, all scalar potential contributions to the elements of the superscripted $Z_n$'s in [5, Eq. (6)] must be suppressed from the calculation of the elements of $Z_n^{mm}$, $Z_n^{em}$, and $Z_n^{me}$. Otherwise, severe roundoff error would occur for small k. Because the elements of the superscripted $Z_n$'s in (86) become proportional to k or $k^2$ when k is small, machine underflows will occur if k is too small. In an attempt to avoid underflows, the elements of the superscripted $Z_n$'s in (86) were divided by $(\frac{1}{2} k\Delta_1)^2$. However, this normalization only decreases the value of k at which underflows begin to occur. Unfortunately, as k approaches zero, the increasing disparity between the real and imaginary parts of the matrix elements eventually exceeds the dynamic range of the computer.

## VI. NUMERICAL RESULTS

The new E-field solution was used to calculate the electric current $\underline{J}$ and electric charge $q_e$ induced by a plane wave axially incident on a conducting circular disk of radius 0.002λ and a conducting sphere of radius 0.002λ. The magnitudes of these currents and charges are presented here. The new E-field solution was also used to obtain the current and charge on a conducting disk of radius $10^{-15}\lambda$ and a conducting sphere of radius $10^{-15}\lambda$. These currents and charges are not shown here. It suffices to state that they agreed well with the known currents and charges on a small disk [14] and a sphere [15, Eq. (6-103)]. The new E-field solution could not be calculated for disks and spheres of radii considerably less than $10^{-15}\lambda$ because of machine underflows.

The conducting disk lies in the xy plane. On the disk, the incident field $(\underline{E}^{inc}, \underline{H}^{inc})$ is given by

$$\underline{E}^{inc} = \underline{u}_x \eta \quad (106)$$

$$\underline{H}^{inc} = - \underline{u}_y \quad (107)$$

where $\underline{u}_x$ and $\underline{u}_y$ are the unit vectors in the x and y directions, respectively. In this case, $\underline{J}$ can be expressed as

$$\underline{J} = \underline{u}_t J_t \cos \phi + \underline{u}_\phi J_\phi \sin \phi \quad (108)$$

where $J_t$ and $J_\phi$ are dimensionless functions of t. In (108), t is the distance from the center of the disk, $\phi$ is the azimuthal angle, $\underline{u}_t$ is the unit vector in the radial direction, and $\underline{u}_\phi$ is the unit vector perpendicular to $\underline{u}_t$. The electric charge $q_e$ can be expressed as

$$q_e = \frac{q}{c} \cos \phi \quad (109)$$

where c is the speed of light and q is a dimensionless function of t.

Figure 1 shows $|J_t|$ on the disk of radius 0.002λ. The symbols × represent the new E-field solution for $|J_t|$. These symbols are plotted at the center of the disk, at the peaks of the triangle functions $\{T_j(t)\}$ in (82), and at the rim of the disk. The solid curve is the known solution for $|J_t|$. This curve was obtained by calculating the known solution at 31 points equally spaced in t and by drawing straight lines between them. The first of these points is at the center of the disk. The last point is at the rim of the disk.

Figure 2 shows $|J_\phi|$ on the disk. The symbols × represent the new E-field solution for $|J_\phi|$. These symbols are plotted at the centers of the pulse functions $\{P_j(t)\}$ in (83). The solid curve is the known solution for $|J_\phi|$. This curve was obtained by calculating the known solution at all of the 31 points mentioned in the previous paragraph except the point at the rim and by drawing straight lines between them.

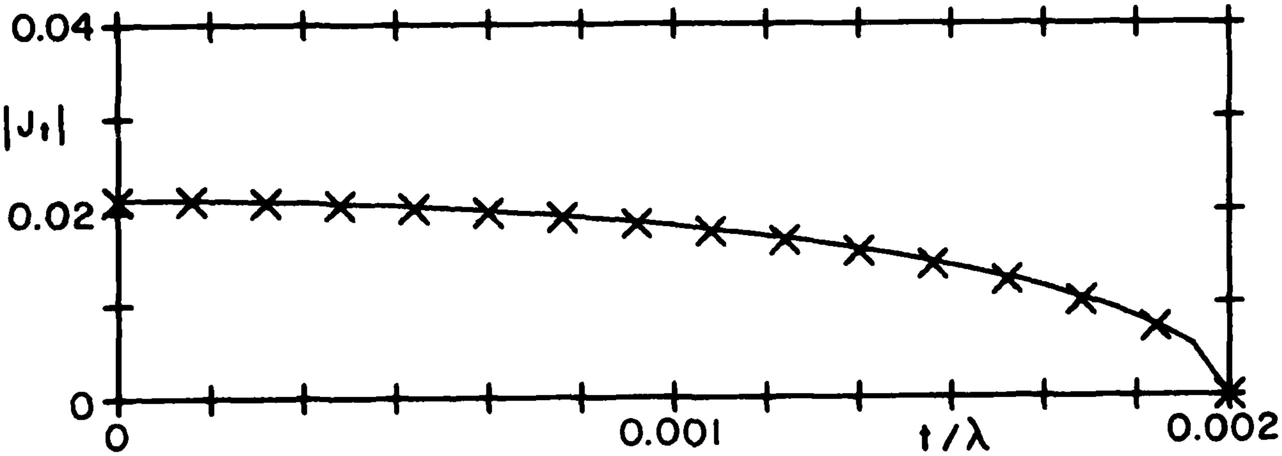


Fig. 1. $|J_t|$ on the conducting circular disk of radius $0.002\lambda$. The symbols $\times$ represent the new E-field solution. The solid curve is the known solution.

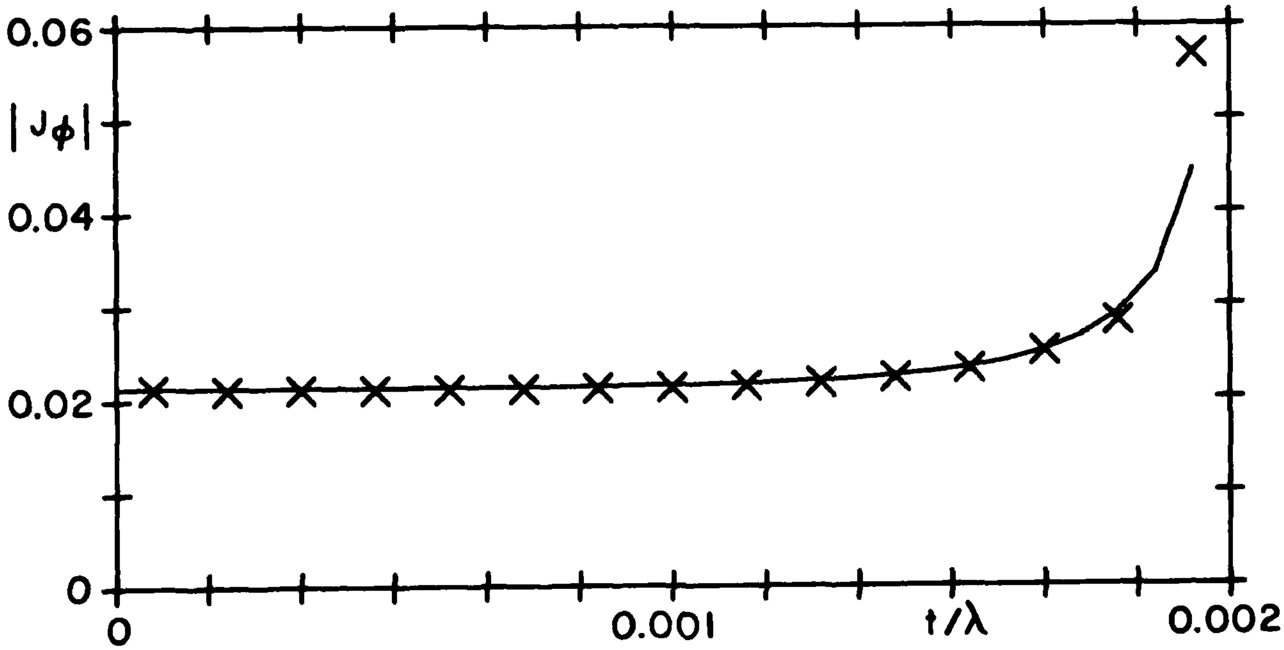


Fig. 2. $|J_\phi|$ on the conducting circular disk of radius $0.002\lambda$.
The symbols $\times$ represent the new E-field solution.
The solid curve is the known solution.

The known solution for $|J_\phi|$ approaches infinity as the reciprocal of the square root of the distance from the rim. Figure 3 shows $|q|$ on the disk in the same way that Fig. 2 shows $|J_\phi|$.

In Fig. 2, the × nearest the rim is 28% higher than the known value of $|J_\phi|$ at the corresponding point. In Fig. 3, the × nearest the rim is 36% higher than the known value of $|q|$ at the corresponding point. These kinds of errors are to be expected when pulses are used to expand a function that goes to infinity as the reciprocal of the square root of the distance from an edge [18].

According to (107), $\underline{H}^{inc}$ has no component normal to the disk. Therefore, the magnetostatic current $\underline{J}^{(0)}$ vanishes so that the electric current $\underline{J}$ reduces to $k\underline{J}^{(1)}$ for small k. In Section IV, it was shown that the new E-field solution for $\underline{J}$ approaches $\underline{J}^{(0)}$ as k approaches zero. It was not shown that if $\underline{J}^{(0)}$ vanishes, then the new E-field solution will reduce to $k\underline{J}^{(1)}$. The good agreement of the new E-field solution for $\underline{J}$ with the known $\underline{J}$ in Figs. 1 and 2 is fortunate. It is a pleasure to state that the new E-field solution for $\underline{J}$ agreed just as well with the known $\underline{J}$ on the disk of radius $10^{-15}\lambda$.

The conducting sphere is placed at the origin and is illuminated by the incident field

$$\underline{E}^{inc} = \underline{u}_x \eta \, e^{jkz} \tag{110}$$

$$\underline{H}^{inc} = - \underline{u}_y \, e^{jkz} \tag{111}$$

The electric current $\underline{J}$ and electric charge $q_e$ induced on the sphere are expressed by (108) and (109), respectively. For the sphere of radius $0.002\lambda$, Fig. 4 shows $|J_t|$, Fig. 5 shows $|J_\phi|$, and Fig. 6 shows $|q|$. On

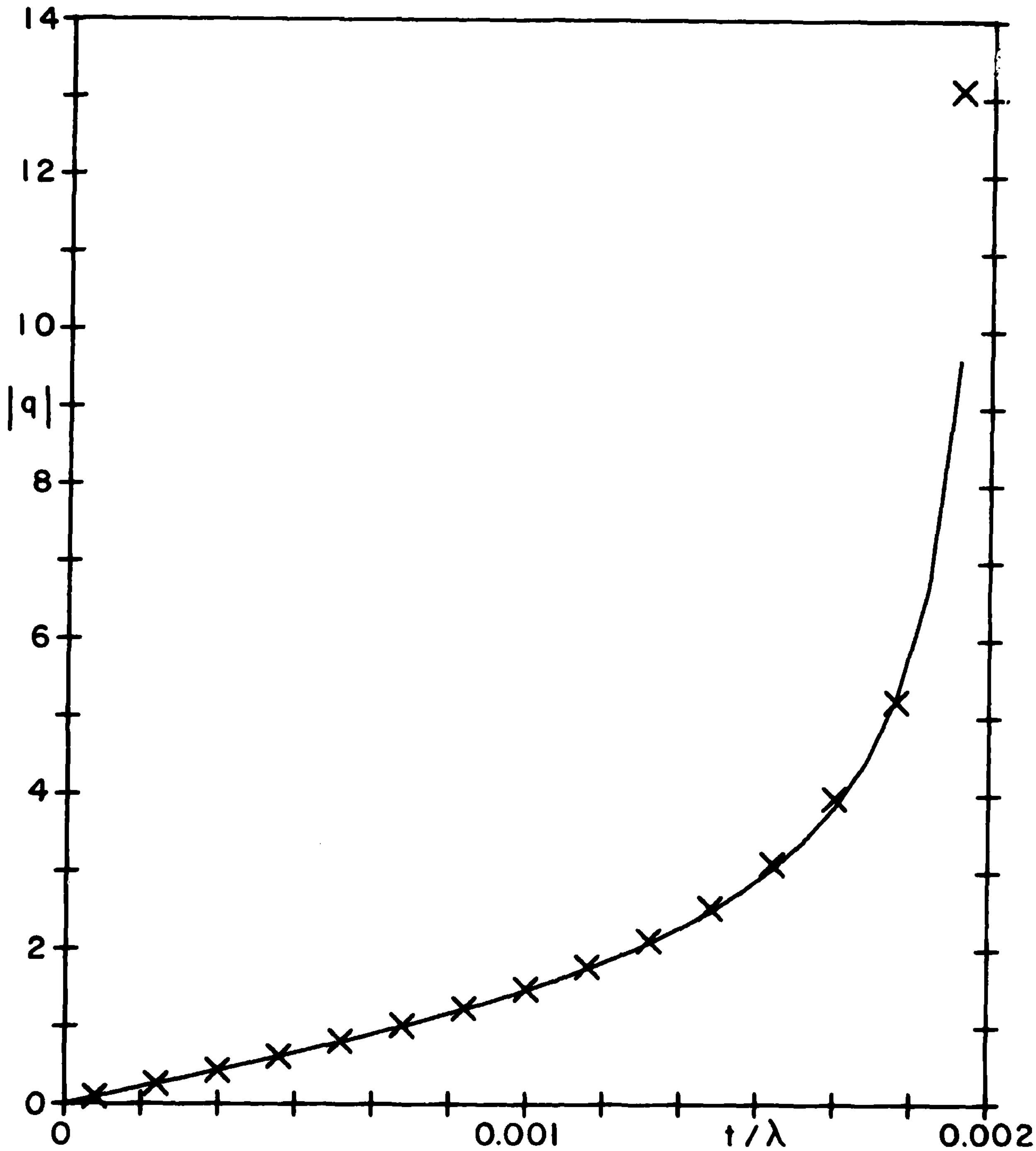


Fig. 3. |q| on the conducting circular disk of radius 0.002λ. The symbols × represent the new E-field solution. The solid curve is the known solution.

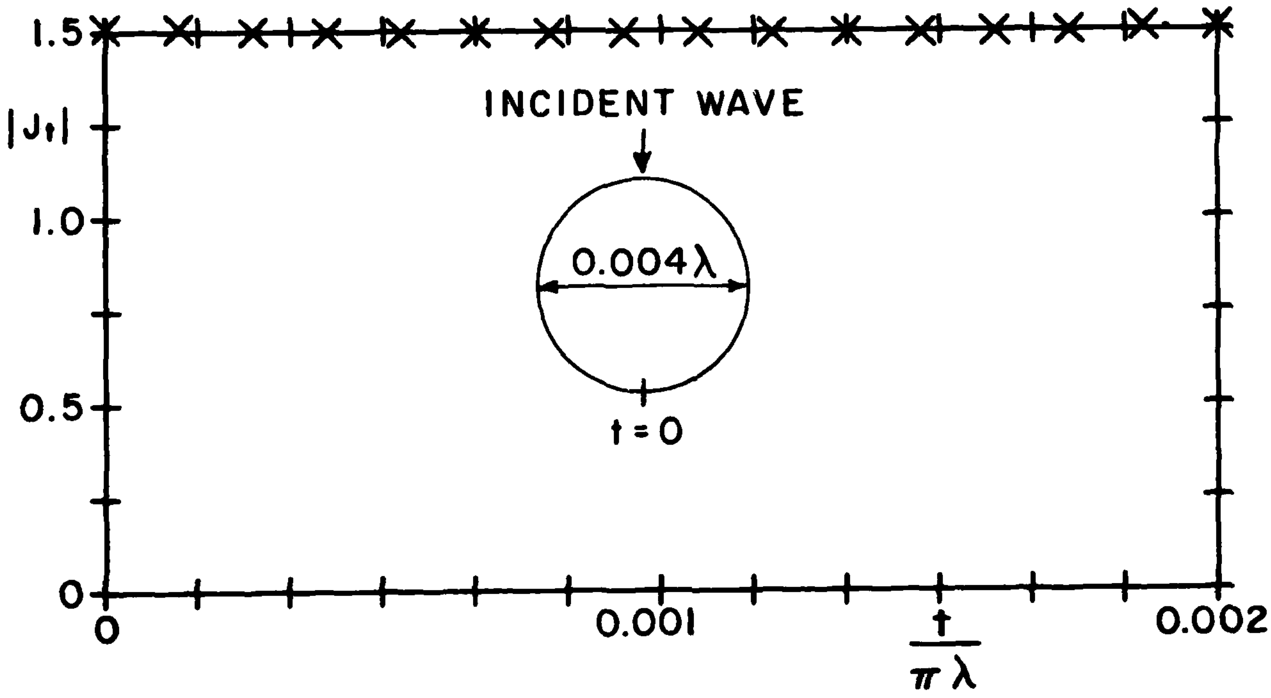


Fig. 4. $|J_t|$ on the conducting sphere of radius 0.002λ. The symbols × represent the new E-field solution. The solid curve is the known solution.

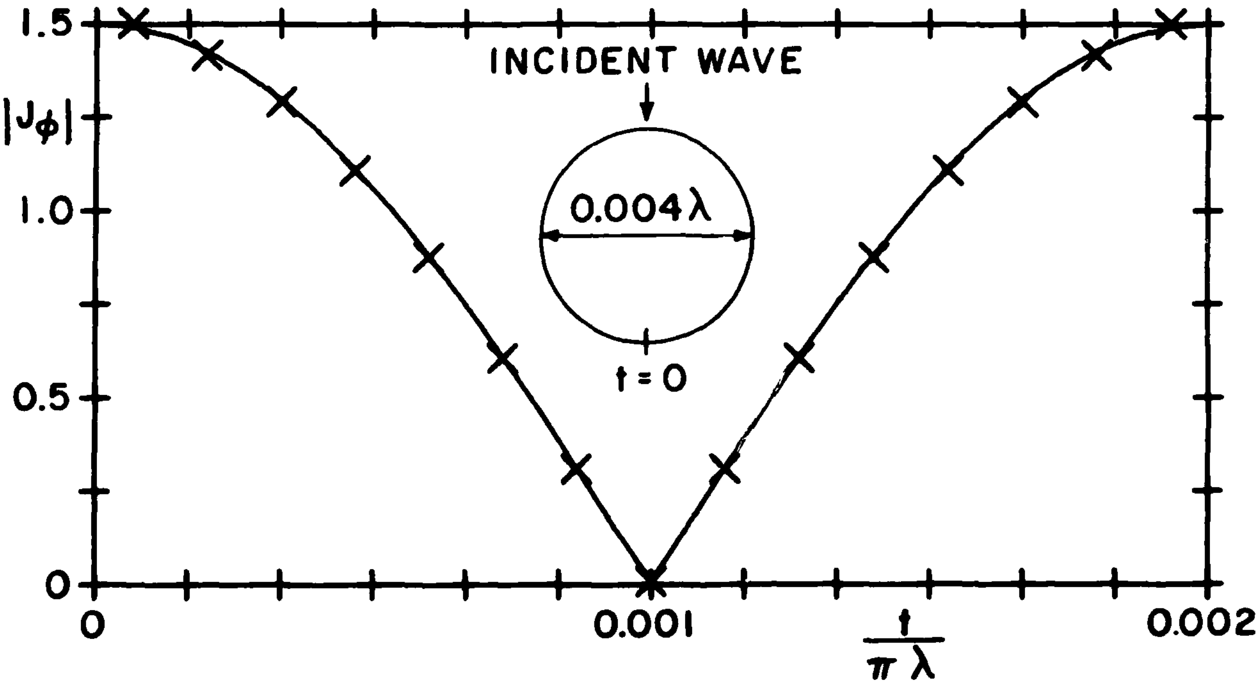


Fig. 5. $|J_\phi|$ on the conducting sphere of radius 0.002λ. The symbols × represent the new E-field solution. The solid curve is the known solution.

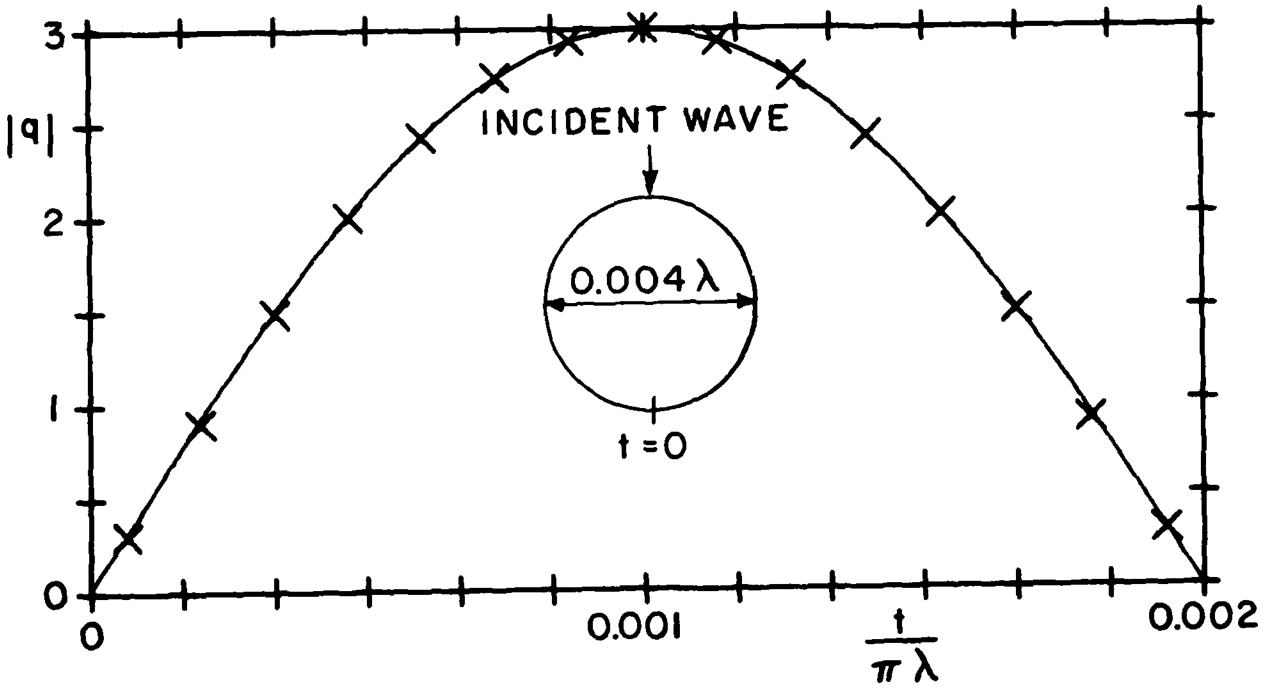


Fig. 6. $|q|$ on the conducting sphere of radius $0.002\lambda$. The symbols $\times$ represent the new E-field solution. The solid curve is the known solution.

this sphere, t is zero at (x=y=0, z=−0.002λ) and is 0.002πλ at (x=y=0, z=0.002λ). As in Figs. 1, 2, and 3, the symbols × in each of Figs. 4, 5, and 6 represent the new E-field solution and the solid curve is the known solution.

# APPENDIX A

In Appendix A, it is shown that

$$\iint_S \underline{W} \cdot \nabla\phi \, ds = - \iint_S \phi(\nabla_s \cdot \underline{W}) ds + \int_C \psi(\underline{W} \cdot \underline{u}_b) d\ell \qquad \text{(A-1)}$$

where S is a finite surface, ds is the differential element of area, $\nabla_s \cdot$ is the surface divergence on S, $\phi$ is a differentiable scalar function defined in 3-dimensional space, and $\underline{W}$ is a differentiable vector function defined on S. Furthermore, $\underline{W}$ is tangent to S. The surface S may be either open or closed. If S is closed, then the second term on the right-hand side of (A-1) is to be omitted. If S is open, then S has an edge. This edge consists of one or more closed contours and is called C. In the second term on the right-hand side of (A-1), $d\ell$ is the differential element of length along C and $\underline{u}_b$ is a unit vector tangent to S and normal to C. For definiteness $\underline{u}_b$ is taken to point away from S.

The following reasoning is used to show that (A-1) is true. Because $\underline{W}$ is tangent to S, the divergence theorem [17, Eq. (42) on p. 503] for $\phi\underline{W}$ is

$$\iint_S \nabla_s \cdot (\phi\underline{W}) ds = \begin{cases} 0 & , \quad \text{S closed} \\ \int_C \phi(\underline{W} \cdot \underline{u}_b) d\ell & , \quad \text{S open} \end{cases} \qquad \text{(A-2)}$$

From [17, Eq. (18) on p. 501],

$$\nabla_s \cdot (\phi\underline{W}) = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial v_1} (h_2 \phi W_1) + \frac{\partial}{\partial v_2} (h_1 \phi W_2)\right] \qquad \text{(A-3)}$$

where $(v_1, v_2)$ are orthogonal curvilinear coordinates on S, $(h_1, h_2)$ are the corresponding metrical coefficients, and $(W_1, W_2)$ are, respectively, the components of $\underline{W}$ in the directions of increasing $v_1$ and $v_2$. Differentiating the products on the right-hand side of (A-3), we obtain

$$\nabla_s \cdot (\phi \underline{W}) = \phi \nabla_s \cdot \underline{W} + \underline{W} \cdot \nabla_s \phi \tag{A-4}$$

where

$$\nabla_s \cdot \underline{W} = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial v_1} (h_2 W_1) + \frac{\partial}{\partial v_2} (h_1 W_2)\right] \tag{A-5}$$

$$\nabla_s \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial v_1} \underline{u}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial v_2} \underline{u}_2 \tag{A-6}$$

Here, $\underline{u}_1$ and $\underline{u}_2$ are, respectively, the unit vectors in the $v_1$ and $v_2$ coordinate directions. According to [17, Eq. (18) on p. 501], the right-hand side of (A-5) is indeed the surface divergence of $\underline{W}$. From [17, Eq. (17) on p. 501], the right-hand side of (A-6) is the surface gradient of $\phi$.

Substitution of (A-4) into (A-2) gives

$$\iint_S \underline{W} \cdot \nabla_s \phi \, ds = - \iint_S \phi (\nabla_s \cdot \underline{W}) ds + \int_C \phi (\underline{W} \cdot \underline{u}_b) d\ell \tag{A-7}$$

Because $\nabla_s$ is the component of $\nabla$ tangent to S and because $\underline{W}$ is tangent to S,

$$\underline{W} \cdot \nabla_s \phi = \underline{W} \cdot \nabla \phi \qquad \text{on S} \tag{A-8}$$

Substitution of (A-8) into (A-7) gives the desired result (A-1).

# APPENDIX B

In Appendix B, it is shown that any differentiable vector function $\underline{W}(\underline{r})$ which is tangent to a surface S and which has no surface divergence can be written as

$$\underline{W}(\underline{r}) = \underline{n} \times \nabla_s \phi(\underline{r}) \; , \quad \underline{r} \text{ on S} \qquad \text{(B-1)}$$

where $\underline{n}$ is the unit vector normal to S, $\nabla_s$ is the surface gradient on S, and

$$\phi(\underline{r}) = \phi(\underline{r}_o) - \int_c (\underline{n}' \times \underline{W}(\underline{r}')) \cdot d\underline{r}' \; , \quad \underline{r} \text{ on S} \qquad \text{(B-2)}$$

In (B-2), $\underline{r}_o$ is the position vector of an arbitrary point on S, c is any contour on S from $\underline{r}_o$ to $\underline{r}$, and $\underline{n}'$ is $\underline{n}$ evaluated at $\underline{r}'$.

The following reasoning is used to show that (B-1) is true. The surface divergence of $\underline{W}$ is called $\nabla_s \cdot \underline{W}$ and is defined by [17, Eq. (18) on p. 501]

$$\nabla_s \cdot \underline{W} = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial v_1} (h_2 W_1) + \frac{\partial}{\partial v_2} (h_1 W_2)\right] \qquad \text{(B-3)}$$

where $(v_1, v_2)$ are orthogonal curvilinear coordinates on S, and $(h_1, h_2)$ are the corresponding metrical coefficients. Also, $W_1$ is the component of $\underline{W}$ in the direction of increasing $v_1$, and $W_2$ is the component of $\underline{W}$ in the direction of increasing $v_2$. Since $\underline{W}$ has no surface divergence,

$$\frac{\partial}{\partial v_1} (-h_2 W_1) = \frac{\partial}{\partial v_2} (h_1 W_2) \qquad \text{(B-4)}$$

In view of (B-4), the differential form

$$h_1 W_2 dv_1 - h_2 W_1 dv_2 \qquad \text{(B-5)}$$

is exact. Therefore, there is a scalar function $\phi(v_1, v_2)$ such that

$$h_1 W_2 = \frac{\partial}{\partial v_1} \phi(v_1, v_2) \tag{B-6}$$

$$-h_2 W_1 = \frac{\partial}{\partial v_2} \phi(v_1, v_2) \tag{B-7}$$

From (B-6) and (B-7), we obtain

$$\underline{W} = -\frac{1}{h_2}\frac{\partial\phi}{\partial v_2}\underline{u}_1 + \frac{1}{h_1}\frac{\partial\phi}{\partial v_1}\underline{u}_2 \tag{B-8}$$

where $(u_1, u_2)$ are, respectively, the unit vectors in the directions of increasing $v_1$ and $v_2$. If the unit vectors $(\underline{u}_1, \underline{u}_2, \underline{n})$ form a right-handed orthogonal system, then (B-8) can be rewritten as

$$\underline{W} = \underline{n} \times \nabla_s \phi \tag{B-9}$$

where

$$\nabla_s \phi = \frac{1}{h_1}\frac{\partial\phi}{\partial v_1}\underline{u}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial v_2}\underline{u}_2 \tag{B-10}$$

According to [17, Eq. (17) on page 501], the right-hand side of (B-10) is indeed the surface gradient of $\phi$. Hence, (B-9) coincides with (B-1).

It is evident from (B-8) and (B-10) that

$$\nabla_s \phi = -\underline{n} \times \underline{W} \tag{B-11}$$

The desired result (B-2) follows from (B-11).

## APPENDIX C

It is shown in Appendix C that

$$\iint_S (\underline{n} \times \nabla_s \phi) \cdot \underline{E}\, ds = -\iint_S \phi(\nabla \times \underline{E}) \cdot \underline{n}\, ds + \int_C \phi(\underline{E} \cdot \underline{u}_\ell) d\ell \qquad \text{(C-1)}$$

where S is a finite surface, ds is the differential element of area, $\underline{n}$ is the unit vector normal to S, $\nabla_s$ is the surface gradient on S, $\phi$ is a differentiable scalar function defined on S, and $\underline{E}$ is a differentiable vector function defined in 3-dimensional space. The surface S may be either open or closed. If S is closed, then the second term on the right-hand side of (C-1) is to be omitted. If S is open, then S has an edge. This edge consists of one or more closed contours and is called C. In the second term on the right-hand side of (C-1), $\underline{u}_\ell$ is the unit vector tangent to C and $d\ell$ is the differential element of length along C. The direction of $\underline{n}$ is the direction that a right-handed screw would advance when turned in the direction of $\underline{u}_\ell$.

The following reasoning is used to show that (C-1) is true. Stokes' theorem [17, Eq. (42) on p. 489] for $\phi \underline{E}$ is

$$\iint_S \underline{n} \cdot \nabla \times (\phi \underline{E})\, ds = \begin{cases} 0 & , \quad \text{S closed} \\ \int_C \phi(\underline{E} \cdot \underline{u}_\ell) d\ell & , \quad \text{S open} \end{cases} \qquad \text{(C-2)}$$

If $(v_1, v_2)$ are orthogonal coordinates on S and if $(\underline{u}_1, \underline{u}_2, \underline{n})$ form a right-handed system where $(\underline{u}_1, \underline{u}_2)$ are, respectively, the unit vectors in the directions of increasing $v_1$ and $v_2$, then [17, Eq. (166) on p. 497]

$$\underline{n} \cdot \nabla \times (\phi\, \underline{E}) = \frac{1}{h_1 h_2} \left(\frac{\partial}{\partial v_1} (h_2\, \phi\, E_2) - \frac{\partial}{\partial v_2} (h_1\, \phi\, E_1)\right) \quad \text{on } S \qquad \text{(C-3)}$$

In (C-3), ($h_1$, $h_2$) are, respectively, the metrical coefficients associated with $v_1$ and $v_2$, and ($E_1$, $E_2$) are, respectively, the components of $\underline{E}$ in the $\underline{u}_1$ and $\underline{u}_2$ directions. Differentiating the products on the right-hand side of (C-3), we obtain

$$\underline{n} \cdot \nabla \times (\phi\, \underline{E}) = \phi(\nabla \times \underline{E}) \cdot \underline{n} + (\underline{n} \times \nabla_s \phi) \cdot \underline{E} \quad \text{on } S \qquad \text{(C-4)}$$

where

$$\nabla_s \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial v_1} \underline{u}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial v_2} \underline{u}_2 \qquad \text{(C-5)}$$

According to [17, Eq. (17) on p. 501], the right-hand side of (C-5) is indeed the surface gradient of $\phi$.

Substitution of (C-4) into (C-2) gives the desired result (C-1).

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